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I. Joseph

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Ilon Joseph*
Lawrence Livermore National Laboratory
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* joseph5@llnl.gov

I. INTRODUCTION

Magneto-hydrodynamics (MHD) critically relies on viscous forces in order for an accurate determination of the electric field. For each charged particle species, the Braginskii viscous tensor for a magnetized plasma has the decomposition into matrices with special symmetries:

$$\boldsymbol{\pi} = \boldsymbol{\pi}_c + \boldsymbol{\pi}_g + \boldsymbol{\pi}_d \tag{1}$$

representing the Chew-Goldberger-Low (CGL) parallel anisotropy $\boldsymbol{\pi}_c$, the gyroviscosity $\boldsymbol{\pi}_g$, which is a completely collisionless effect, and dissipation $\boldsymbol{\pi}_d$ perpendicular to the magnetic field $\vec{B}$. In the following, indices representing particle species (e, i corresponding to electrons, ions) will be suppressed for clarity unless explicitly necessary.

The results are simplified by consideration of the lowest order in finite Larmor radius (FLR) effects, flute-reduction, and in nonlinearity. The small parameter is $\delta \sim k_\parallel/k_\perp \sim k_\parallel \rho \sim \omega/\Omega \sim \delta n/n$ where Ω is the gyrofrequency and ρ is the gyroradius, k refers to characteristic spatial wavenumbers and ω refers to characteristic frequencies. We will also simplify by assuming low $\beta = 2\mu_0 p/B^2 \ll 1$ and assume $\beta \sim \mathcal{O}(\delta)$, where p is the pressure and μ_0 is the permeability of free space. Furthermore, we simplify by assuming that the current $\mu_0 \vec{J} = \vec{\nabla} \times \vec{B}$ is $\mathcal{O}(\beta)$ relative to the field. This is certainly true for the perpendicular current, but is also a good approximation for the parallel current in most situations of interest. Finally, the flute reduction consists of the assumption of incompressible perpendicular motion $\vec{\nabla} \cdot n\vec{v}_\perp \simeq 0$ and that the electric field satisfies $\vec{E}_\perp = -\vec{\nabla}_\perp \phi$.

To explain the importance of the results, we describe the case where the parallel flows are negligible. We will find that the usual vorticity equation is modified by the CGL viscous force to the following form

$$\varpi = \vec{\nabla} \cdot \Omega_i^{-2} Z_i e (Z_i e n_i \vec{\nabla}_\perp \phi + \vec{\nabla}_\perp p_i) \tag{2}$$

$$(\partial_t + \vec{\nabla} \cdot \vec{v}_E)\varpi = -\vec{\nabla} \cdot J_\parallel \hat{b} + 2\hat{b} \times \vec{\nabla} B \cdot \vec{\nabla}\left(p_{tot} + 5\pi_{ci}/6\right)/B^2. \tag{3}$$

The last term represents the modification by the CGL viscous force. We will find that a simple approximation for the CGL anisotropy coefficients is

$$\pi_{cj} = \eta_{0j} \vec{v}_{\perp j} \cdot \vec{\nabla} \log B \tag{4}$$

where $\eta_{0j} \propto p_j/\nu_{\mathrm{eff,j}}$ and the effective collision frequency is $\nu_{\mathrm{eff,j}}$. This causes a damping of the vorticity proportional to

$$\nu_\varpi = (k_Z v_{th}/k_\perp R)^2/\nu_{\mathrm{eff}} \tag{5}$$

where $k_Z = -i\hat{b} \times \hat{\kappa} \cdot \vec{\nabla}$. Here, the velocity is defined to 1st order in the FLR expansion via

$$\vec{v} = v_\parallel \hat{b} + \vec{v}_\perp \tag{6}$$

$$\vec{v}_\perp = \vec{v}_E + \vec{v}_{pj} \tag{7}$$

$$\vec{v}_E = \vec{E} \times \hat{b}/B \tag{8}$$

$$\vec{v}_{pj} = \hat{b} \times \vec{\nabla} p_j/m\Omega. \tag{9}$$

and $\Omega = ZeB/m$ is the gyrofrequency.

The effective collision frequency $\nu_{\mathrm{eff,j}}$ varies from the usual collision frequency ν_c in the collisional Pfirsch-Shlüter regime to ω_t^2/ν_c in the low-collisionality banana regime. Here, we will focus on the collisional regime, but modifications at low collisionality can be simply obtained with a suitable (typically Padé) approximation for $1/\nu_{\mathrm{eff}}$.

II. REDUCED MHD EQUATIONS

The viscous force density $\vec{F}_\pi = -\vec{\nabla} \cdot \boldsymbol{\pi}$ can be decomposed into parallel and perpendicular components. The parallel component of the force is

$$F_{\pi\parallel} = -\hat{b} \cdot \vec{\nabla} \cdot \boldsymbol{\pi} = \boldsymbol{\pi} : \vec{\nabla}\hat{b} - \vec{\nabla} \cdot \vec{\pi}_\parallel. \tag{10}$$

Using the gyro-viscous cancellation, the parallel force balance equation is

$$\partial_t mnv_\parallel + \vec{\nabla} \cdot mn(v_\parallel^2 + \vec{v}_E v_\parallel) - mnv_\parallel \mathbf{v} \cdot \vec{\nabla}\hat{b} = -\partial_\parallel p_{tot} + F_{\pi\parallel}. \tag{11}$$

Here we have used Ohm's law to eliminate the electric field. Since the electron viscous force is smaller than the ion viscous force by $(m_e/m_i)^{1/2}$ only the ion viscous force needs to be retained.

The perpendicular component of the force drives a polarization current via the divergence

$$\vec{\nabla}\cdot\frac{\hat{b}}{B}\times\vec{F}_\pi = \vec{F}_\pi\cdot\vec{\nabla}\times\frac{\hat{b}}{B} - \frac{\hat{b}}{B}\cdot\vec{\nabla}\times\vec{F}_\pi \tag{12}$$

$$= \vec{F}_\pi\cdot\vec{\nabla}B^{-2}\times\vec{B} + \vec{F}_\pi\cdot\mu_0\vec{J}/B^2 - \frac{\hat{b}}{B}\cdot\vec{\nabla}\times\vec{F}_\pi. \tag{13}$$

The second term is $\mathcal{O}(\beta)$ and can be neglected relative to the others.

The flute reduction implies that the perpendicular flow is almost incompressible. The electric potential can then be determined by the curl of the perpendicular force balance equation. In order to eliminate the electric field from this equation a sum over all charged species must be performed. Thus, the full definition of the vorticity involves a sum over all species

$$\varpi = \vec{\nabla}\cdot\sum_j \Omega_j^{-2} Z_j e(Z_j e n_i \vec{\nabla}_\perp\phi + \vec{\nabla}_\perp p_j). \tag{14}$$

Since each term is proportional to m_j, the electron contribution can be neglected in this expression. To lowest order the gyro-viscous cancellation between the diamagnetic flow and the gyro-viscous tensor implies that

$$\left(\partial_t + \vec{\nabla}\cdot\vec{v}_E\right)\varpi = -\vec{\nabla}\cdot J_\parallel\hat{b} + \vec{\nabla}\cdot\frac{\hat{b}}{B}\times\vec{F}_* \tag{15}$$

where $\vec{v}_E = \vec{E}\times\vec{B}/B^2$ and, in this equation, $\vec{F}_*$ is the sum of all forces aside from the $\vec{J}\times\vec{B}$ force and the gyro-viscous force.

Again, Ohm's law can be used to eliminate the electric field. Only the ion viscous forces needs to be retained since the electron viscous forces are smaller by the factor $(m_e/m_i)^{1/2}$. To lowest order, the contribution of the total pressure is $\vec{\nabla}\cdot\frac{\hat{b}}{B}\times\vec{\nabla}p_{tot} = 2\hat{b}\times\vec{\nabla}B\cdot\vec{\nabla}p_{tot}/B^2$. (See Appendix A for how to simplify calculations involving the magnetic field.) The goal of the next section is to calculate the contribution of the CGL viscous force.

III. CGL VISCOSITY

A. CGL Forces

The Chew-Goldberger-Low viscous tensor $\boldsymbol{\pi}_c$ is "gyrotropic" meaning that it only allows anisotropy along the axis of the magnetic field $\hat{b}$. It is also traceless because the trace of the gyrotropic pressure tensor is absorbed in the definition of the pressure $p = (p_\parallel + 2p_\perp)/3$. Thus, the CGL tensor has the form

$$\boldsymbol{\pi}_c = \pi_c\left(\hat{b}\hat{b} - \mathbb{I}/3\right). \tag{16}$$

The CGL force is explicitly

$$\vec{F}_c = \vec{\nabla}\pi_c/3 - \vec{\nabla}\cdot\pi_c\hat{b}\hat{b} = \vec{\nabla}\pi_c/3 - \vec{B}\cdot\vec{\nabla}\pi_c\hat{b}/B \tag{17}$$

$$= \vec{\nabla}\pi_c/3 - \vec{B}\partial_\parallel\pi_c/B - \pi_c\vec{\kappa}. \tag{18}$$

The parallel component can be rewritten as

$$F_{c\parallel} = -\pi_c\vec{\nabla}\cdot\hat{b}/3 - \vec{\nabla}\cdot 2\pi_c\hat{b}/3 \tag{19}$$

$$= \boxed{-B^{3/2}\partial_\parallel 2\pi_c/3B^{3/2}.} \tag{20}$$

Vorticity is driven by

$$\vec{\nabla}\cdot\frac{\hat{b}}{B}\times\vec{F}_c = \vec{\nabla}\cdot\frac{\hat{b}}{B}\times\vec{\nabla}\frac{\pi_c}{3} - \vec{\nabla}\cdot\frac{\hat{b}}{B}\times\pi_c\vec{\kappa} \tag{21}$$

$$= \vec{B}\cdot\left(\vec{\nabla}\frac{\pi_c}{3}\times\vec{\nabla}B^{-2} - \vec{\kappa}\times\vec{\nabla}\frac{\pi_c}{B^2}\right) + \frac{\pi_c}{B}\hat{b}\cdot\vec{\nabla}\times\vec{\kappa} + \frac{\mu_0\vec{J}}{B}\cdot\left(\vec{\nabla}\frac{\pi_c}{3} - \pi_c\vec{\kappa}\right). \tag{22}$$

Neglecting terms of order $\mathcal{O}(\beta)$ yields

$$\boxed{\vec{\nabla}\cdot\frac{\hat{b}}{B}\times\vec{F}_c \simeq \hat{b}\cdot\vec{\nabla}\log B\times\vec{\nabla}\frac{5\pi_c}{3B}.} \tag{23}$$

B. Simplified Parallel Anisotropy

Here we follow the notation of Simakov and Catto [1], but use a slightly simplified anisotropy in order to simplify the discussion. The changes to the results are discussed at the end of this section.

In the linear drift-MHD (1st order FLR) regime, the CGL stress tensor can be defined in terms of the following rate of strain tensor formed from

$$\boldsymbol{\alpha} = \partial_i v_j + \frac{2}{5p}\partial_i q_j \tag{24}$$

$$\mathbf{W} = \boldsymbol{\alpha} + \boldsymbol{\alpha}^T \tag{25}$$

$$\bar{\mathbf{W}} = \mathbf{W} - \frac{2}{3}\vec{\nabla}\cdot\vec{v} - \frac{4}{15p}\vec{\nabla}\cdot\vec{q}. \tag{26}$$

The following definitions automatically generate a traceless form for the viscous tensor and are equivalent whether one uses $\bar{\mathbf{W}}$ or $\mathbf{W}$. To simplify notation below, we also define the vectors

$$\vec{w} = \vec{v} + \frac{2\vec{q}}{5p} \tag{27}$$

Each equation is given per species, but the species indices are suppressed for clarity. The Braginskii viscosity coefficients for each species are

$$\eta_{0,i} = 0.96 p_i/\nu_i \tag{28}$$

$$\eta_{0,e} = 0.73 p_e/\nu_e. \tag{29}$$

The parallel stress anisotropy is defined as

$$\begin{aligned}
\pi_c &= -\eta_0\left(3\hat{b}\hat{b}-\mathbb{I}\right):\mathbf{W}/2 && (30)\\
&= -\eta_0\left(3\hat{b}\cdot\partial_\parallel\vec{v} - \vec{\nabla}\cdot\vec{v} + \frac{6}{5p}\hat{b}\cdot\partial_\parallel\vec{q} - \frac{2}{5p}\vec{\nabla}\cdot\vec{q}\right) && (31)\\
&= -\eta_0\left(3\partial_\parallel v_\parallel + \frac{6}{5p}\partial_\parallel q_\parallel - \vec{\nabla}\cdot\vec{v} - \frac{2}{5p}\vec{\nabla}\cdot\vec{q} - 3\vec{w}_\perp\cdot\vec{\kappa}\right) && (32)\\
&= \boxed{-\eta_0\left(2\partial_\parallel v_\parallel + \frac{4}{5p}\partial_\parallel q_\parallel + w_\parallel\partial_\parallel\log B - \vec{\nabla}\cdot\vec{v}_\perp - \frac{2}{5p}\vec{\nabla}\cdot\vec{q}_\perp - 3\vec{w}_\perp\cdot\vec{\kappa}\right).} && (33)
\end{aligned}$$

The perpendicular flows are determined from the finite Larmor radius (FLR) expansion:

$$\vec{v}_\perp = \frac{\hat{b}}{ZeB}\times\left(n^{-1}\vec{\nabla}p - Ze\vec{E}\right) \tag{34}$$

$$\frac{2\vec{q}_\perp}{5p} = \frac{\hat{b}}{ZeB}\times\vec{\nabla}T \tag{35}$$

$$\vec{w}_\perp = \frac{\hat{b}}{ZeB}\times\left(n^{-1}\vec{\nabla}p + \vec{\nabla}T - Ze\vec{E}\right). \tag{36}$$

Given the flute-reduced electrostatic approximation $\vec{E}_\perp \simeq -\vec{\nabla}_\perp\phi$, the perpendicular divergence is

$$\vec{\nabla}\cdot\vec{v}_\perp \simeq \frac{\vec{B}}{Ze}\cdot\left(\vec{\nabla}p\times\vec{\nabla}n^{-1}B^{-2} - Ze\vec{E}\times\vec{\nabla}B^{-2}\right) + \frac{\vec{v}_\perp\cdot\mu_0\vec{J}}{B} \tag{37}$$

$$\frac{2\vec{\nabla}\cdot\vec{q}_\perp}{5p} = \frac{\vec{B}}{Zep}\cdot\vec{\nabla}T\times\vec{\nabla}pB^{-2} + \frac{2\vec{q}_\perp\cdot\mu_0\vec{J}}{5pB} \tag{38}$$

$$\vec{\nabla}\cdot\vec{v}_\perp + \frac{2\vec{\nabla}\cdot\vec{q}_\perp}{5p} = \vec{B}\cdot\left(T\vec{\nabla}\log n + 2n\nabla T - Ze\vec{E}\right)\times\vec{\nabla}B^{-2} + \frac{\vec{w}_\perp\cdot\mu_0\vec{J}}{B}. \tag{39}$$

The final terms involving J are $\mathcal{O}(\beta)$ and can be neglected relative to the others. The combination is approximately

$$\vec{\nabla}\cdot\vec{v}_\perp + \frac{2\vec{\nabla}\cdot\vec{q}_\perp}{5p} - 3\vec{w}_\perp\cdot\vec{\kappa} \simeq -\vec{v}_\perp\cdot\vec{\nabla}\log B. \tag{40}$$

To lowest order in δ, β the result is

$$\boxed{\pi_c = -\eta_0\left(2B^{-1/2}\partial_\parallel B^{1/2}v_\parallel + \frac{4}{5p}B^{-1/2}\partial_\parallel B^{1/2}q_\parallel - \vec{v}_\perp\cdot\vec{\nabla}\log B\right).} \tag{41}$$

According to Ref. [1], there are some additional corrections to this form. The exact modifications can be found in these references. For the perpendicular viscous force that enters the vorticity equation, this leads to a modified $\bar{v}_\perp = v_\perp + \xi\hat{b}\times\vec{\nabla}T/m\Omega$ where

$$\xi_i = 0.68 \tag{42}$$

$$\xi_e = 0.98. \tag{43}$$

Thus, the temperature gradient driven component of the drift is enhanced by the factor $(1+\xi)$.

C. Viscous Timescales

For simplicity, we now assume that the temperature is constant and seek to determine the viscous timescales. If one were to consider parallel flow $v_\parallel$ only, then the evolution to the state $\pi_c \propto B^{-1/2}$ occurs on a consistently ordered timescale $(k_\parallel v_t)^2/\nu \sim \mathcal{O}(\delta)$. On the other hand, we can neglect the parallel velocity and consider the vorticity equation alone with the anisotropy

$$\pi_c = \eta_0\vec{v}_\perp\cdot\vec{\nabla}\log B. \tag{44}$$

This will only vanish iff the (ion) species are electrostatically confined so that $Zen\vec{E} = \vec{\nabla}p$ along the direction orthogonal to $\vec{\nabla}B$. The size of $\vec{\nabla}\pi/L_B$ term is $\sim k_\perp^2\rho^2\omega_t^2/\nu\Omega$ where $\omega_t = v_t/L_B$. This should be compared to the vorticity equation which is of order $\sim (k_\perp\rho)^2\omega/\Omega$. Thus, as long as $\omega_t^2/\nu\Omega \sim \mathcal{O}(\delta)$, the equations are consistently ordered.

Appendix A: Magnetic Field Calculations

The magnetic field **B** and current $\mu_0\vec{J} = \vec{\nabla}\times\vec{B}$ are related to the magnetic curvature $\vec{\kappa}$ via

$$\vec{\kappa} = \hat{b}\cdot\vec{\nabla}\hat{b} \tag{A1}$$

$$\mu_0\vec{J}\times\vec{B} = \vec{\kappa}B^2 - \vec{\nabla}_\perp B^2/2. \tag{A2}$$

For an MHD equilibrium

$$\vec{J}\times\vec{B} = \vec{\nabla}p_{tot} \tag{A3}$$

$$\vec{\kappa}B^2 = \mu_0\vec{\nabla}p_{tot} + \vec{\nabla}_\perp B^2/2. \tag{A4}$$

Clearly $\mu_0 J_\perp/B \sim \mathcal{O}(\beta)$ and we will simplify by further assuming that $\mu_0 J_\parallel/B \sim \mathcal{O}(\beta)$ as well.

We will need an expression for

$$\vec{\nabla}\times\frac{\hat{b}}{B} = \vec{\nabla}B^{-2}\times\mathbf{B} + \frac{\mu_0 J}{B^2} \tag{A5}$$

$$= \frac{\hat{b}}{B}\times(\vec{\nabla}\log B + \vec{\kappa}) + \frac{\mu_0 J_\parallel\hat{b}}{B^2}. \tag{A6}$$

For an MHD equilibrium

$$\vec{\nabla}\times\frac{\hat{b}}{B} = \frac{\hat{b}}{B}\times(2\vec{\nabla}\log B + B^{-2}\vec{\nabla}\mu_0 p_{tot}) + \frac{\mu_0 J_\parallel\hat{b}}{B^2}. \tag{A7}$$

Neglecting terms $\sim \mathcal{O}(\beta)$ yields

$$\boxed{\vec{\nabla} \times \frac{\hat{b}}{B} \simeq \frac{\hat{b}}{B} \times 2\vec{\nabla} \log B \simeq \frac{\hat{b}}{B} \times 2\kappa}. \tag{A8}$$

We will now show that $\hat{b} \cdot \vec{\nabla} \times \vec{\kappa} \sim \mathcal{O}(\beta)$:

$$\vec{\kappa} \simeq \vec{\nabla}_\perp \log B \tag{A9}$$

$$\vec{\nabla} \times \vec{\kappa} \simeq -\vec{\nabla} \times \frac{\hat{b}}{B} \partial_\| B = -\vec{\nabla} \frac{\partial_\| B}{B^2} \times \vec{B} - \frac{\mu_0 \vec{J}}{B} \partial_\| \log B \tag{A10}$$

$$\simeq -\vec{\nabla} \frac{\partial_\| B}{B^2} \times \vec{B}. \tag{A11}$$

For an MHD equilibrium, the result is clearly $\mathcal{O}(\beta)$:

$$\vec{\nabla} \times \vec{\kappa} = -\vec{\nabla} \frac{\partial_\| B}{B^2} \times \vec{B} - \mu_0 \vec{J} \frac{\partial_\| B}{B^2} + \vec{\nabla} B^{-2} \times \vec{\nabla} \mu_0 p \tag{A12}$$

$$\hat{b} \cdot \vec{\nabla} \times \vec{\kappa} = -\frac{\mu_0 J_\|}{B} \partial_\| \log B + \hat{b} \cdot \vec{\nabla} B^{-2} \times \vec{\nabla} \mu_0 p. \tag{A13}$$

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[1] A. N. Simakov and P. J. Catto, Phys. Plasmas **10**, 4744 (2003); A. N. Simakov and P. J. Catto, Phys. Plasmas **11**, 2326 (2004).